Introduction to spherical geometry

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1 Introduction

One of two standard non euclidean geometries. Hyperbolic geometry became fashionable because Thurston started it. Spherical geometry is still rather dormant.

There are analogies between hyperbolic and spherical geometries.

1.1 Transitional geometry

Continuous passage between spherical and hyperbolic geometry, containing in the middle Euclidean geometry. Thurston talked about the transition between 8 geometries in dimension 3.

2 Basics of spherical geometry

In dimension 2, think of S^2 in $\mathbb{R}^3$. Need to specify lines and triangles, and trigonometric formulae. The equator is a line in the sphere. More generally, a line is an intersection of a plane in $\mathbb{R}^3$ with the sphere. If the plane passes through the origin, then the line is a great circle. If the two planes defining the line meet somewhere, the angle between the lines is the angle between the planes. If we now take three lines, we get a triangle bounded by the lines. This is the object of interest in spherical geometry. In hyperbolic geometry, triangles are defined similarly.

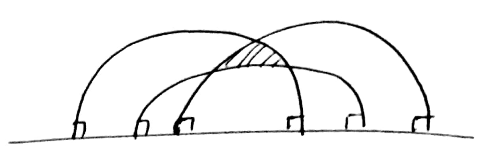

Figure 1: Example of a triangle in hyperbolic geometry

Lengths of line segments on the sphere is defined as the angle they subtend at the centre.

2.1 Trigonometric formulae

We have the following formula in spherical geometry.

$$\frac{\sin BC}{\sin \hat{A}} = \frac{\sin AB}{\sin \hat{C}} = \frac{\sin AC}{\sin \hat{B}} \tag{1}$$

Similarly, we have a formula for hyperbolic geometry.

$$\frac{\sinh BC}{\sin \hat{A}} = \frac{\sinh AB}{\sin \hat{C}} = \frac{\sinh AC}{\sin \hat{B}} \tag{2}$$

In euclidean geometry, we have the following.

$$\frac{BC}{\sin \hat{A}} = \frac{AB}{\sin \hat{C}} = \frac{AC}{\sin \hat{B}} \tag{3}$$

Another interesting formula is the following. Take four lines through a point. Cut them with two other lines. Then in euclidean geometry, we have this.

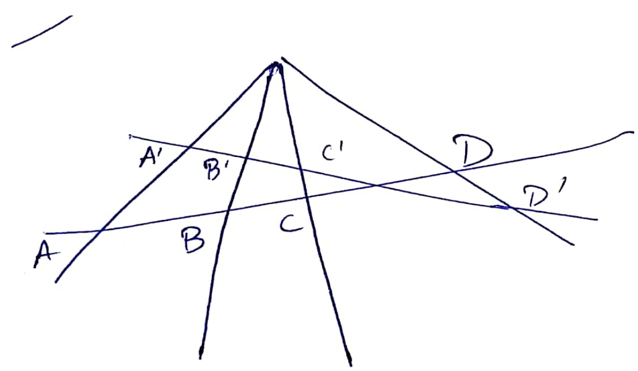

Figure 2: The cross ratio result

$$\frac{BD}{CD} \cdot \frac{CA}{BA} = \frac{B'D'}{C'D'} \cdot \frac{C'A'}{B'A'} \tag{4}$$

In spherical geometry, we have a similar formula.

$$\frac{\sin BD}{\sin CD} \cdot \frac{\sin CA}{\sin BA} = \frac{\sin B'D'}{\sin C'D'} \cdot \frac{\sin C'A'}{\sin B'A'} \tag{5}$$

In hyperbolic space, the formula is same, but the sin replaced with sinh.

2.2 Triangle inequalities

Euclid shows on the intersection of three planes, the three dihedral angles satisfy triangle inequality.

Proposition 2.1. *If* ABC *is a triangle, then* $AB + BC \geqslant AC$*.*

Proposition 2.2 (Isosceles Triangle)**.** $\hat{A} = \hat{C} \equiv BA = BC$.

Proposition 2.3 (Congruence theorems)**.** *If the three sides are equal, then the triangles are isometric. If two sides and the angle between them is equal, then the triangles are isometric.*

Proposition 2.4 (Comparison theorem)**.** *If two sides are same, and one angle is bigger, then the third side is bigger.*

2.3 Results unique to spherical geometry

Proposition 2.5. *If two triangles have the same angles, they are isometric.*

Lemma 2.6 (The Lemma)**.** *Given a triangle* ABC*, with an exterior angle* BCD*. There are three cases*

1. $AB + BC$ *is a semicircle (i.e. angle subtended is* π*) iff* $B\hat{C}D = \hat{A}$*.*
2. $AB + BC < \pi$ *iff* $B\hat{C}D > \hat{A}$*.*
3. $AB + BC > \pi$ *iff* $B\hat{C}D < \hat{A}$*.*

Sketch of proof. If $AB + BC = \pi$, then $BC = BD$. This implies $B\hat{C}D = B\hat{D}C$. But $\hat{D} = \hat{A}$. This proves the result. The other parts follow from the comparison lemma. □

Proposition 2.7. *We have a triangle* ABC *with exterior angle* $B\hat{C}D$*. Then* $B\hat{C}D < \hat{A} + \hat{B}$*.*

Proof. Construct an angle $D\hat{C}E = \hat{A}$. But The Lemma tells us that $AE + EC = \pi$. And this tells us that $BE + EC < \pi$. Then $\hat{B} > B\hat{C}E$. Adding $\hat{A}$ to both sides, we get the following.

$$\hat{A} + \hat{B} > B\hat{C}D \tag{6}$$

□

Corollary 2.8. *The sum of angles in a triangle is greater than* π*.*

In hyperbolic geometry, the reverse inequality holds, i.e. sum of angles in a triangle is less than π.

Definition 2.1 (Angle excess)**.** The angle excess of a triangle ABC is $\hat{A} + \hat{B} + \hat{C} - \pi$.

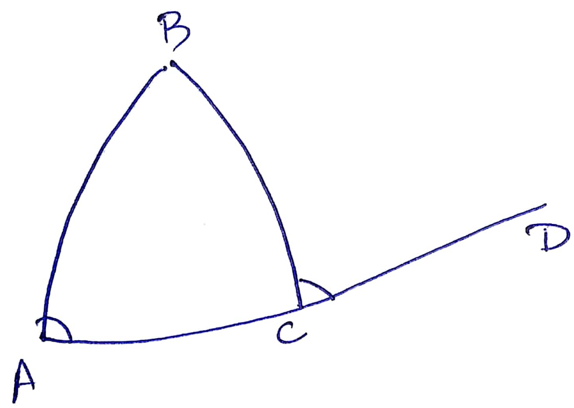

Figure 3: The exterior angle

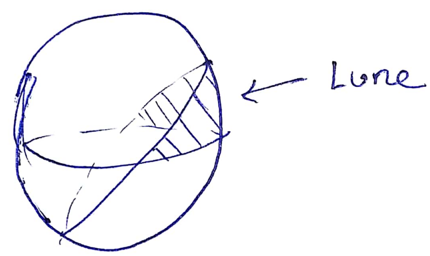

Figure 4: A lune

2.4 Area

Observation: Suppose you divide a triangle into two triangles ABC and ACF. The sum of the angle excess of the two triangles is the angle excess of the larger triangle. Up to a constant, the area must be the angle excess. A similar result also holds in hyperbolic geometry.

Proposition 2.9. *Suppose there exists a triangle* ABC*. Let* D *and* E *be the midpoints of* A *and* BC[1]. *Then* $DE > \frac{1}{2}AC$.

Remark: This is often the definition of positively curved manifolds.
There a more precise expression.

$$\cos DE = \cos\left(\frac{AC}{2}\right) \cdot \cos\left(\frac{1}{2}|ABC|\right) \tag{7}$$

[1] Called the Busemann positive curvature..

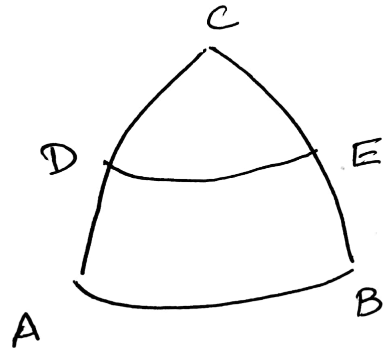

Figure 5: Midpoint inequality

Proof of theorem 2.9. Extend ED to DG such that $ED = GD$[2]. Now extend AG to intersect

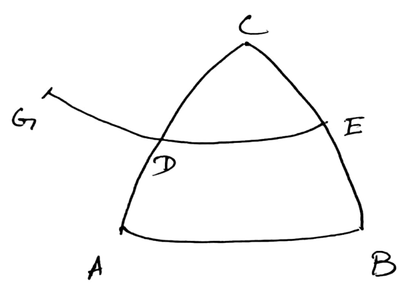

Figure 6: Extending ED to EG.

BC at H. We have that $GDA = EDC$. By The Lemma, we have $CH + AH = \pi$. This implies $EH + HA > \pi$. This gives us that $B\hat{E}A < H\hat{A}E$.

We now consider the triangles AEB and AEG. The have a side in common, and two sides equal, and one angle larger than other, which means $GE > AB$. But $GE = 2 \cdot DE$, which gives us $DE > \frac{1}{2}AB$. □

We also have the following expressions.

$$\cos DE = \cos\left(\frac{1}{2}|ABC|\right) \cdot \cos\left(\frac{AB}{2}\right) \tag{8}$$

Euler's formula for area is given by the following.

$$\cos\left(\frac{1}{2}|ABC|\right) = \frac{1 + \cos a + \cos b + \cos c}{4\cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)} \tag{9}$$

[2]We assume the triangle edges are less than a semicircle.

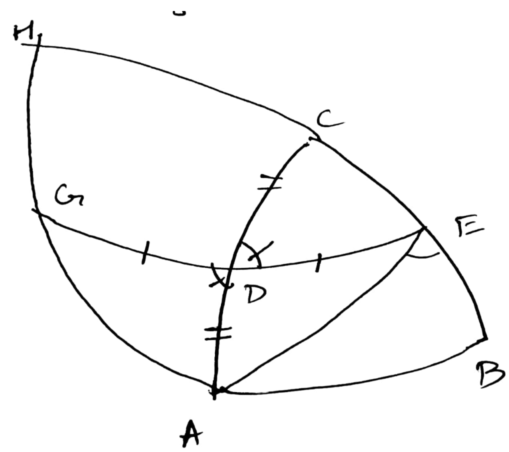

Figure 7: Intersecting AG and BC at H.

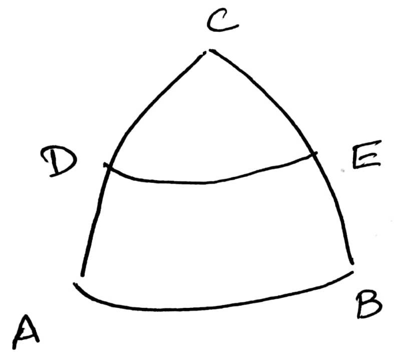

Figure 8: Midpoint inequality

2.5 Median of a triangle in spherical geometry

Consider a triangle. It's a fact that the line joining the three midpoints intersect at one point. In euclidean geometry, we have $AO = 2OD$. In spherical geometry, we have $AO < 2OD$. In hyperbolic geometry, $AO > 2OD$[3].

Proposition 2.10 (Critera for lines to meet)**.** *If we have an equilateral triangle as below, and the following conditions are satisfied, then the lines meet at a point.*

$$2 = \frac{\tan AO}{\tan OD} = \frac{\tan BO}{\tan OE} = \frac{\tan CO}{\tan OF} \tag{10}$$

The proof for this is due to Euler. The problem for the general case is still open.

[3]Euler knew only for equilateral triangle. In fact Euler had Ceva's theorem for spherical equilateral triangles.

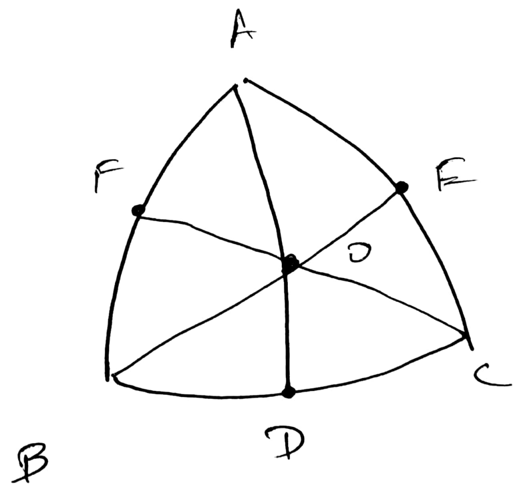

Figure 9: The three lines intersecting at a point.

3 Convexity

Definition 3.1 (Convex set)**.** A subset Ω of the sphere is said to be convex if for any two points, the shortest path joining them lies in Ω.

Such a set must necessarily lie in Ω.
Consider the spherical cross ratio $[A, B, C, D]$.

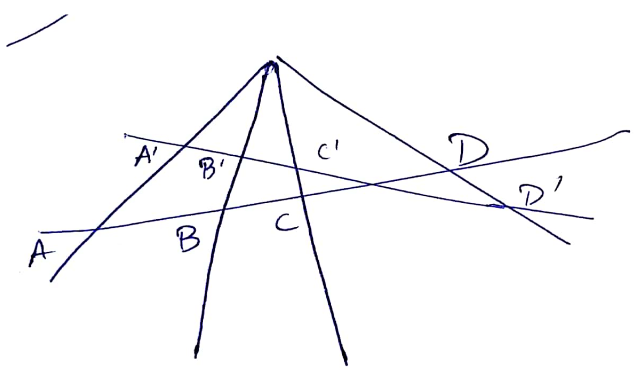

Figure 10: Cross ratio

$$[A, B, C, D] = \frac{\sin BD}{\sin CD} \cdot \frac{\sin CA}{\sin BA} \tag{11}$$

In the Cayley-Klein model for hyperbolic geometry, the distance is given by the following formula.

$$d(A, B) = \frac{1}{2} \log[A', A, B, B'] \tag{12}$$

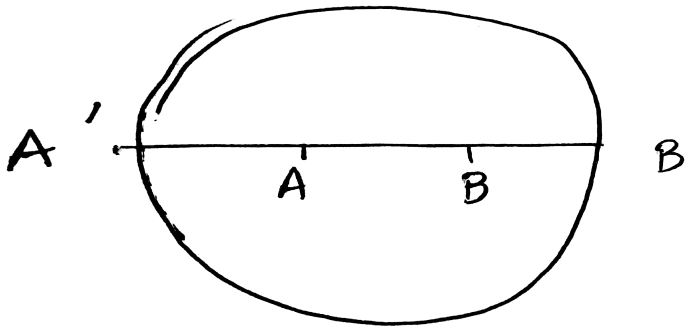

Figure 11: Hyperbolic metric in $\mathbb{B}^n$

We can do this metric construction on any convex subset Ω of S^2. Similar questions can be asked in this setting[4].

3.1 Funk geometry

We work in a similar setting. We defin $F(A, B)$ as the following.

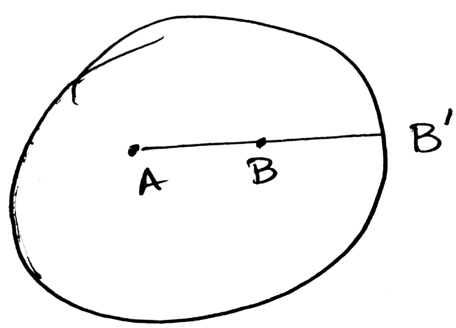

Figure 12: Funk metric

$$F(A, B) = \log \frac{AB'}{BB'} \tag{13}$$

It's not symmetric, but it satisfies the other criteria for metrics. Symmetrizing this metric gives you the Hilbert metric. On convex sets in S^2, we thus have the Funk metric as well.

[4]Hilbert Geometry.

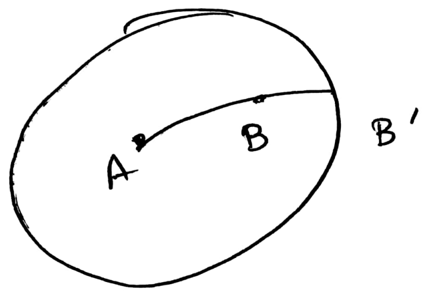

Figure 13: Funk metric in S^2

$$F(A,B) = \log \frac{\sin AB'}{\sin BB'} \tag{14}$$

3.2 Hilbert Geometry on the sphere

In this setting, we can ask questions like if there are parallel lines or not. It's easy to see the euclidean segments are the geodesics for Hilbert metric.

Definition 3.2 (Geodesic). A geodesic is a path $[0,1] \to \Omega$ such that for all $t_1 \leqslant t_2 \leqslant t_3$, we have the following.

$$F(g(t_1), g(t_2)) + F(g(t_2), g(t_3)) = F(g(t_1), g(t_3)) \tag{15}$$

Hilbert was interested in metrics in which euclidean lines are geodesics. Hilbert's Fourth Problem asks for the a characterization of metrics on which euclidean lines were geodesics[5][6].

Now consider a sphere minus two convex subsets. We can define a distance between two points A and B as the following.

$$H(A,B) = \log[A', A, B, B'] \tag{16}$$

This is called a "time-like geometry". Each point in this geometry has a future and a past cone.

[5]Problem still open for non-symmetric metrics.

[6]Busemann-Phadke worked on "distinguished geodesics".

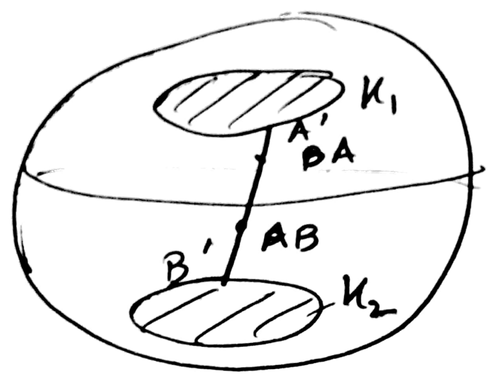

Figure 14: Metric on a sphere S^2 with K_1 and K_2 removed.