The Laplacian on Riemannian Manifolds

Linking analysis on manifolds to their geometry

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Outline

What is the Laplacian?

- On open subsets of $\mathbb{R}^n$, the Laplacian is a second order partial differential operator.

$$\Delta f = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)(f)$$

- Succinctly expressed by the following expression.

$$\Delta f = \mathrm{div}(\mathrm{grad}(f))$$

Why do we care about the Laplacian?

- Harmonic functions satisfy properties similar to holomorphic functions.
  - Regularity
  - Mean value property
- General heuristic that elliptic operators provide insight into global topology of the space.

How do we define the Laplacian on manifolds?

- Need a coordinate independent definition.
- Can't use the formula involving the sums of second partial derivatives.
- To define Δ using div and grad, we need a Riemannian metric.
- Using a Riemannian metric, we get notions analogous to grad and div.
  - grad generalizes to exterior derivative operator d.
  - div generalizes to codifferential operator δ.

Definition

The Laplacian acting on differential forms on a Riemannian manifold (M, g) is given by the following expression.

$$\Delta\alpha = d\delta\alpha + \delta d\alpha$$

Statement of the Hodge decomposition theorem

- For which $\alpha \in E^p$ does the following PDE have a solution?

$$\Delta\omega = \alpha$$

- Only for α orthogonal to the space of harmonic forms.

Theorem (Hodge decomposition theorem)

The space of smooth p-forms E^p has the following orthogonal direct sum decomposition.

$$E^p = \Delta(E^p) \oplus H^p$$

Furthermore, the space H^p is finite-dimensional.

Idea of proof

- Easy to show $\Delta(E^p) \subseteq (H^p)^\perp$.
- To show $(H^p)^\perp \subseteq \Delta(E^p)$, we need to solve the following PDE for $\alpha \in (H^p)^\perp$.

$$\Delta\omega = \alpha$$

- We look for *weak* solutions γ.

$$\gamma(\Delta\beta) = \langle \alpha, \beta \rangle$$

- Using technical results, show weak solution is smooth.

Technical results

Theorem (Precompactness lemma)

If α_i is a sequence of smooth p-forms such that there is constant C bounding the L^2-norm of all the elements of the sequence $\{\alpha_i\}$ and $\{\Delta\alpha_i\}$, then the sequence $\{\alpha_i\}$ has a Cauchy subsequence.

Theorem (Elliptic regularity)

Let γ be a weak solution to $\Delta\omega = \alpha$. Then there exists a smooth p-form η_γ such that $\gamma(\beta) = \langle \eta_\gamma, \beta \rangle$. In particular, that means that η_γ is a solution to $\Delta\omega = \alpha$.

Relation to cohomology

Theorem

The linear map π *that projects closed p-forms to harmonic p-forms is surjective and* $\pi(\alpha) = \pi(\beta)$ *iff* α *and* β *differ by an exact form. This means the* p^{th} *De Rham cohomology space is isomorphic to the space of harmonic p-forms.*

$$\mathcal{H}^p_{\mathrm{DR}}(M) \cong H^p(M)$$

Application: The Bochner technique

- The Bochner technique relies on identities of the following kind.

$$(\text{Term involving } \Delta\alpha) = (\text{A non-negative term}) + (\text{Term involving curvature})$$

- A prototypical example for 1-forms.

$$\int \langle \Delta\alpha, \alpha \rangle = \int |\nabla\alpha|^2 + \int \mathrm{Ric}\left(\alpha^\sharp, \alpha^\sharp\right)$$

- Positive definite Ricci curvature implies first cohomology vanishes.

Some eigenfacts about the Laplacian

- The eigenvalues of the Laplacian are all non-negative.
- The eigenvalues form a discrete subset of $[0, \infty)$.

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

Can we estimate λ_1 in terms of the geometric properties of the manifold?

Lower bound on λ_1

Theorem

If M is a compact manifold whose Ricci curvature is bounded below by κ, i.e. the Ricci curvature satisfies the following inequality for all vector fields η.

$$\mathrm{Ric}(\eta, \eta) \geq \kappa(n-1)\,|\eta|^2$$

Then λ_1 satisfies the following inequality.

$$\lambda_1 \geq n\kappa$$

Proved using yet another Bochner style identity.

$$\Delta\left(|\mathrm{grad}\ f|^2\right) = |\mathrm{Hess}\ f|^2 + \langle \mathrm{grad}\ f, \mathrm{grad}\ \Delta f\rangle + \mathrm{Ric}(\mathrm{grad}\ f, \mathrm{grad}\ f)$$

Linking Hessian and Laplacian

- Hessian is a map from TM to TM.

$$(\mathrm{Hess}\ f)(\eta) = \nabla_\eta(\mathrm{grad}\ f)$$

- The Laplacian can be obtained from the Hessian.

$$\Delta f = \mathrm{tr}(\mathrm{Hess}\ f)$$

- Inequality linking Frobenius norm of operator to trace.

$$n|T|^2 \geq \mathrm{tr}(T)^2$$