Constructing complex structures on surfaces via the Beltrami equation

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Abstract

Coming up with concrete examples of Riemann surfaces of genus 2 or higher is not very easy. One way to do so is to start with an orientable Riemannian manifold of dimension 2, and look at the conformal class of that metric, and that canonically corresponds to a complex structure on the surface. In this talk, we'll see how to get the complex structure from the conformal structure by solving the Beltrami equation.

1 The problem with holomorphic maps

Much like how a smooth manifold is defined in a first course on the subject, one can define Riemann surfaces in an analogous manner.

Definition 1.1 (Riemann surface)**.** A Riemann surface X is a topological space which is Hausdorff and second countable such that every point $x \in X$ has an open set U around it, and a homeomorphism ϕ to an open subset of $\mathbb{C}$ such that for any other choice of homeomorphism ϕ' to an open subset of $\mathbb{C}$, $\phi' \circ \phi^{-1}$ is a holomorphic function.

For all purposes, it seems like a fairly innocuous definition, and just a generalization of the definition of a smooth surface. But anyone who has taken a complex analysis course knows well that holomorphic functions are often annoyingly rigid. Unlike smooth functions, you can't just construct a holomorphic function to satisfy any property you want: for instance, the only holomorphic function that vanishes in an open set is the constant function 0. This means that you have no chance at all of constructing bump functions on Riemann surfaces.

In fact, we are faced with a more fundamental question. Barring obvious examples like $\mathbb{C}$ and $\widehat{\mathbb{C}}$, do we have any examples of Riemann surfaces? One way to get those would to quotient out $\mathbb{C}$, and open subsets of it by isometries that are free, properly discontinuous, and act without fixed points. Finding such isometries of $\mathbb{C}$ is not too hard, and quotienting by those give us the complex tori. Can we do better? Can we actually get surfaces of higher genus? The approach involving isometries might not be easy, and will probably involve some hard algebra.

Another possible approach is to start off with smooth surface, and try to see what sort of obstructions come up, and if we manage to resolve them, we perhaps might end up with a Riemann surface.

2 Turning a smooth surface into a Riemann surface

The first obstruction we come up against is orientability. On a Riemann surface, the change of coordinates map is holomorphic, and the determinant of the Jacobian of a holomorphic map is always non-negative. That means a Riemann surface is an orientable surface, and if we want to turn a surface into a Riemann surface, we need to start with an orientable surface.

While Riemann surfaces don't have a notion of length, they do have a notion of angle between tangent vectors at a point, which is called a conformal structure. That means we can try putting a conformal structure on a surface before we put a complex structure. A way of getting a conformal structure is by constructing a Riemannian metric on the surface, which gives us angles and length of tangent vectors. This is easy to do so because every manifold has a smooth Riemannian metric, which follows from a partition of unity argument.

Now we have a surface with a Riemannian metric, and an associated conformal structure. We would now like to upgrade this conformal structure to a complex structure. We will do so in two steps: we first show that under an additional condition on the conformal structure, it can be upgraded to a complex structure. Then we show that the additional hypothesis we needed to upgrade the conformal structure are actually satisfied by every conformal structure.

The additional hypothesis on the conformal structure we need is that around every point, we can find an open set which is conformally equivalent to an open subset of $\mathbb{R}^2$ with the standard conformal structure. In terms of the Riemannian metric, this hypothesis says that around every point p, we can find coordinates such that the Riemannian metric ds^2 looks like the following.

$$ds^2 = \phi(x, y)(dx^2 + dy^2)$$

Here, ϕ is a positive smooth function on an open set around p. Such a coordinate chart is called an isothermal coordinate chart[1]. If we have isothermal coordinates around each point, with the additional condition that (dx, dy) forms an ordered basis, the claim is that we can upgrade this conformal structure to a complex structure. To do so, we assign the complex coordinate z to be $x + iy$, where (x, y) are the ordered isothermal coordinates. We need to show that this forms a complex structure, i.e. the transition maps are holomorphic. Suppose around a point we have two different isothermal coordinates, call them $z = x + iy$ and $w = a + ib$. To show holomorphicity, it suffices to check the Cauchy-Riemann equation.

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$$
$$-\frac{\partial b}{\partial x} = \frac{\partial a}{\partial y}$$

We also have the tangent vectors in the old and new coordinates.

$$\frac{\partial}{\partial x} = \frac{\partial a}{\partial x}\left(\frac{\partial}{\partial a}\right) + \frac{\partial b}{\partial x}\left(\frac{\partial}{\partial b}\right)$$
$$\frac{\partial}{\partial y} = \frac{\partial a}{\partial y}\left(\frac{\partial}{\partial a}\right) + \frac{\partial b}{\partial y}\left(\frac{\partial}{\partial b}\right)$$

The fact that (x, y) form an isothermal coordinate system tells us that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are orthogonal and have the same norm. Using the same facts for $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$, we get the following equalities.

$$\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial x}\right)^2 = \left(\frac{\partial a}{\partial y}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 \tag{1}$$
$$\left(\frac{\partial a}{\partial x}\right)\left(\frac{\partial a}{\partial y}\right) = \left(\frac{\partial b}{\partial x}\right)\left(\frac{\partial b}{\partial y}\right) \tag{2}$$

And from the fact that our surface was orientable, and (dx, dy) and (da, db) were positively oriented bases, we get an inequality.

$$\left(\frac{\partial a}{\partial x}\right)\left(\frac{\partial b}{\partial y}\right) > \left(\frac{\partial b}{\partial x}\right)\left(\frac{\partial a}{\partial y}\right) \tag{3}$$

Using equations (1), (2), and (3), it's not too hard to conclude that the Cauchy-Riemann equations are satisfied, and we indeed get a complex structure on the manifold. That shows that if we can find isothermal coordinates around every point, we can upgrade the conformal structure to a complex structure.

Now we need to proceed to the second step of our solution: showing that we can always find isothermal coordinates around every point.

[1] It's called an isothermal coordinate chart because in dimension 2, isothermal coordinates are also harmonic, which means $\Delta_g(u_i) = 0$ for all i, or the coordinate functions are steady state solutions of the heat equation. That's where the name isotherm(al) comes from. Source: `https://mathoverflow.net/questions/32169/why-are-they-called-isothermal-coordinates`.

3 Constructing isothermal coordinates

To figure out how to construct isothermal coordinates, it's useful to introduce a new set of coordinates and tangent vectors from a given coordinate system. If we have a coordinate system (x, y), then we can define $z = x + iy$ and $\overline{z} = x - iy$. That gives us a new basis of cotangent vectors: dz and $d\overline{z}$[2]. Now suppose we have a Riemannian metric on the local coordinate system.

$$ds^2 = E dx^2 + 2F dx dy + G dy^2$$

In terms of the dz and $d\overline{z}$, the metric looks like the following.

$$ds^2 = \lambda \left|dz + \mu d\overline{z}\right|^2 \tag{4}$$

The functions λ and μ are the following.

$$\lambda = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right)$$
$$\mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}$$

It might also be helpful to clarify what $|dz + \mu d\overline{z}|^2$ means. In terms of dx and dy, it is the following.

$$|dz + \mu d\overline{z}|^2 = (\mathrm{Re}\,(dx + i dy + \mu(dx - i dy)))^2 + (\mathrm{Im}\,(dx + i dy + \mu(dx - i dy)))^2$$

Now suppose there exists an isothermal coordinate system (u, v). Then the metric looks like the following in the isothermal coordinate system.

$$ds^2 = \rho(u, v)(du^2 + dv^2)$$

The analogue of equation (4) would be the following equation.

$$ds^2 = \rho(w, \overline{w})\,|dw|^2 \tag{5}$$

But we can also write dw in terms of dz and $d\overline{z}$.

$$dw = \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \overline{z}} d\overline{z} \tag{6}$$

Plugging equation (6) back into equation (5), we get the following.

$$ds^2 = \rho(w, \overline{w}) \left|\frac{\partial w}{\partial z}\right|^2 \left|dz + \frac{\frac{\partial w}{\partial \overline{z}}}{\frac{\partial w}{\partial z}} d\overline{z}\right|^2 \tag{7}$$

Comparing equation (7) to equation (4), we see that if we have an isothermal coordinate system w, then w satisfies the following equation.

$$\mu = \frac{\frac{\partial w}{\partial \overline{z}}}{\frac{\partial w}{\partial z}} \tag{8}$$

We rewrite this into the following form.

$$\frac{\partial w}{\partial \overline{z}} + (-\mu)\frac{\partial w}{\partial z} = 0 \tag{9}$$

Conversely, suppose we had a function w with non-zero derivative which satisfied equation (9). Then that would give us an isothermal coordinate system. That means constructing an isothermal coordinate system hinges solely upon finding non-trivial solutions to equation (9). This equation is called the Beltrami equation, and we'll now see why we can find non-trivial solutions for this partial differential equation in our context. This method of reducing the problem to the Beltrami equation is from section 1.5 of [IT12].

[2] I'm not sure what the i in front of y actually does geometrically. All I'm really doing here with dz and $d\overline{z}$ is symbolic manipulation.

4 Solving the Beltrami equation

Notice that all we really need from the Beltrami equation is a local solution around the point 0. That means we can make a number of simplifying assumptions. First we can assume $\mu(0) = 0$. We can do this by choosing the original coordinate system we started out with to be the geodesic normal coordinates centred at 0. That ensures $E(0) = G(0)$ and $F(0) = 0$. We can also now assume that $|\mu| < 0$ by picking a small enough neighbourhood around 0. Furthermore, we can also assume its derivatives, which we'll denote by μ_z and $\mu_{\overline{z}}$ are smaller than ϵ, for some fixed ϵ, by dilating the coordinate z. Finally, we can multiply μ by a bump function to make sure it has compact support. We thus have the following conditions on μ.

(i) $\mu(0) = 0$

(ii) μ is supported on the closed disc of radius $\frac{1}{2}$.

(iii) μ, μ_z and $\mu_{\overline{z}}$ are bounded in size by some fixed $\epsilon > 0$.

In summary, we want to solve the equation $w_{\overline{z}} + \mu w_z$ in some small neighbourhood of 0 for a smooth solution w such that $w_z(0) \neq 0$. Write w as $z + \phi$ to transform the equation to the following form.

$$\phi_{\overline{z}} + \mu\phi_z = -\mu \tag{10}$$

We want to find a smooth solution to equation (10) such that $|\phi_z(0)| < 1$, since that will ensure $w_z(0) \neq 0$.

The first step is solving this PDE is to try to invert the map that sends ϕ to $\phi_{\overline{z}}$. That inverse is given by the following integral operator (called the Beurling transform).

$$(Tu)(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{u(w)}{z-w} dm(w) \tag{11}$$

In the expression above, $dm(w)$ refers to the Lebesgue measure on $\mathbb{C}$ with respect to w. For compactly supported smooth u, the Beurling transform is well defined, and it's not too hard to see that it inverts the operator $\frac{\partial}{\partial \overline{z}}$ (see Exercise 1, Chapter 9 of [Don11]).

$$\frac{\partial}{\partial \overline{z}} Tu = u \tag{12}$$

Since T is invertible, a solution of (10) will look like $T\nu$, for some $\nu \in C_c^\infty$. If we plug this back into (10), and use (12), we end up getting the following.

$$\nu - \mu S\nu = -\mu \tag{13}$$

Here, S is the operator $-\frac{\partial}{\partial z}T$. We can rewrite this as a single operator acting on ν.

$$(1 - \mu S)\nu = -\mu \tag{14}$$

If we can now somehow invert the operator $1 - \mu S$, we'll be done. Naïvely, the inverse of $1 - \mu S$ is the following operator.

$$(1 - \mu S)^{-1} = 1 + \mu S + (\mu S)^2 + (\mu S)^3 + \cdots \tag{15}$$

But it's not clear in what sense does the right hand side converge, if at all. Even if does converge, recall that we want ν to be compactly supported. But applying this operator to μ ensures that what we get is supported in the supported of μ, and since μ is compactly supported, ν will be compactly supported too. Then there's the question of convergence. Recall that we bounded μ, μ_z, and $\mu_{\overline{z}}$ by some unspecified ϵ. This is where that ϵ comes in. If we know what in what sense we want the right hand side to converge, we can make our ϵ small enough to make that convergence happen. The last thing we want to take care of is ensure the result of applying this operator on μ is still a smooth function. For that, it suffices to show the solution we get is differentiable at least once, and because we're dealing with an elliptic operator, elliptic regularity will tell us that the solution is actually smooth.

The right norm in which to consider the convergence of (15) is the Hölder norm. This norm is given by the following expression.

$$[\psi]_{(0,\alpha)} = \sup_x |\psi(x)| + \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^\alpha} \tag{16}$$

Here, α is a fixed constant in $(0, 1)$.

With respect to this norm, S turns out be a bounded operator, and we get one of the *Schauder estimates*.

$$[Su]_{(0,\alpha)} \leqslant K_\alpha [u]_{(0,\alpha)} \tag{17}$$

Here, K_α is a fixed constant depending on α. The proof of this inequality is hard analysis, and can be found in Section 13.3 of [Don11]. If we have the inequality, and we fix an ϵ small enough that $\epsilon K_\alpha < 1$, then our sequence of operators does converge in the Hölder norm. The means we get ν to be the following.

$$\nu = \left(\sum_{i=0}^{\infty} (\mu S)^i \right) (-\mu) \tag{18}$$

But what we were really interested in was $\phi = T\nu$. Which means ϕ is given by the following expression.

$$\phi = \left(\sum_{i=0}^{\infty} (T)(\mu S)^i \right) (-\mu) \tag{19}$$

The only thing we need to verify is that ϕ is differentiable at least once, and $|\phi_z(0)| < 1$. If we differentiate the above expression with respect to $\bar{z}$, we end up getting the expression for ν, which has bounded Hölder norm, and if we differentiate it with respect to z, we end up getting the same sequence shifted by 1, which is still bounded. That shows that the derivatives exist and are bounded, and gives us a bound of $\phi_z(0)$. We can pick a small enough ϵ to make $\phi_z(0)$ smaller than 1, which proves what we set out to prove.

References

[Don11] Simon Donaldson. *Riemann surfaces*. Oxford University Press, 2011.

[IT12] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer Science & Business Media, 2012.